

Finding DFAs with maximal shortest synchronizing word length

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Abstract. It was conjectured by Černý in 1964 that a synchronizing DFA on n states always has a synchronizing word of length $(n-1)^2$, and he gave a class of DFAs for which this bound is reached. In 2007 Trahtman gave an investigation of all DFAs up to certain size for which this bound is reached, and which are not contained in other DFAs with the same smallest synchronizing word length. Here we extend this analysis in two ways: we drop this latter condition, and we drop limits on alphabet size. For $n \leq 4$ we do the full analysis yielding exactly 15 and 12 DFAs for $n = 3, 4$, respectively, up to isomorphism. For $n \geq 5$ we do the analysis assuming Trahtman's conjecture about non-extendable DFAs. In particular, as a main result we prove that the only way in which the Černý examples C_n can be extended keeping the same smallest synchronizing word length $(n-1)^2$, is by copying an existing symbol or adding a symbol that acts as the identity.

1 Introduction

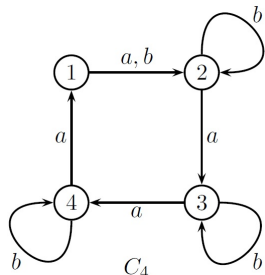
A *deterministic finite automaton (DFA)* over a finite alphabet Σ is called *synchronizing* if admits a *synchronizing word*. Here a word $w \in \Sigma^*$ is called *synchronizing* (or directed, or reset) if starting in any state q , after processing w one always ends in one particular state q_s . So processing w acts as a reset button: no matter in which state the system is, one always jumps to the particular state q_s . Now Černý's conjecture ([3]) states:

Every synchronizing DFA on n states admits a synchronizing word of length $\leq (n-1)^2$.

Surprisingly, despite of extensive effort this conjecture is still open, and even the best known upper bound is still cubic in n . Černý himself ([3]) provided an upper bound of $2^n - n - 1$ for the length of the shortest synchronizing word. A substantial improvement was given by Starke [14], who was the first to give a polynomial upper bound, namely $1 + \frac{1}{2}n(n-1)(n-2)$. The best known upper bound is $\frac{1}{6}(n^3 - n)$, established by Pin in 1983 [11]. He reduced proving this upper bound to a purely combinatorial problem which was then solved by Frankl [8].

Since then for more than 30 years no progress for the general case has been made.

The conjecture has been proved for some particular classes of automata, such as circular automata, aperiodic automata and one-cluster automata with prime length cycle. For these results and some more partial answers, see [1,2,5,6,7,10,12,15,17,18]. For a survey on synchronizing automata and the Černý conjecture, we refer to [19].



In [3] already DFAs were given for which this bound of the conjecture is attained: for $n \geq 2$ the DFA C_n is defined to consist of n states $1, 2, \dots, n$, and two symbols a, b , acting by $\delta(i, a) = i + 1$ for $i = 1, \dots, n - 1$, $\delta(n, a) = 1$, and $\delta(i, b) = i$ for $i = 2, \dots, n$, $\delta(1, b) = 2$. For $n = 4$ this is depicted on the left.

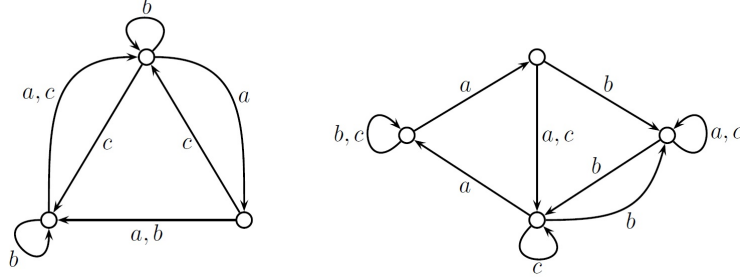
For C_n let $w = b(a^{n-1}b)^{n-2}$, having length $|w| = (n - 1)^2$. It turns out that $qw = 2$ for all $q \in Q$, so w is synchronizing. In [3] it is shown that no shorter synchronizing word exists for C_n , showing that the bound in Černý's conjecture is sharp.

The topic of this paper is to investigate all DFAs for which the bound is reached; these DFAs are called *critical*. A DFA for which the bound is exceeded is called *super-critical*, so Černý's conjecture states that super-critical DFAs do not exist.

An extensive investigation in this direction was already done in [16]: by computer support and clever algorithms all DFAs on n states and q symbols with shortest synchronizing word length $(n - 1)^2$ were investigated for $3 \leq n \leq 7$ and $q \leq 4$, and for $n = 8, 9, 10$ and $q = 2$. Here a minimality requirement was added: examples were excluded if the same property is kept after removing one symbol. Then up to isomorphism there are exactly 8 of them, apart from the basic Černý examples: three with three states, three with four states, one with 5 and one with 6. So apart from the basic Černý examples only a limited number of other DFAs on n states and q symbols with shortest synchronizing word length $(n - 1)^2$ are known, and it is conjectured in [16] that no more exist, a conjecture that we call *Trahtman's conjecture*.

A drawback of the minimality condition is that all solutions are unrelated by definition: no one is contained in another. In order to get a more detailed investigation and also investigate relationship between distinct solutions, in this paper we drop the minimality requirement. So in contrast to [16] we do not exclude examples for which the same property is kept after removing one symbol. In this way one example can be a subset of another one, and distinct minimal examples may share common extensions. Indeed we will see that this occurs. To exclude infinitely many trivial extensions by adding symbols that act exactly as existing symbols, we only consider *basic* DFAs. These are defined to be DFAs in which no two distinct symbols act in the same way in the automaton, and no symbol acts as the identity. Obviously, adding the identity or adding copies of existing symbols has no influence on synchronization. As one main result we prove that up to isomorphism for $n = 3$ there are exactly 15 basic critical DFAs

and for $n = 4$ there are exactly 12 basic critical DFAs. Two of these are the following, showing typical phenomena:



The left one restricted to a, b is exactly C_3 , while restricted to a, c it is exactly a DFA found in [16] that we call T3-1 in Section 3. So this example is a kind of union of C_3 and T3-1. It has four distinct synchronizing words of the minimal length 4: $baab$, $baac$, $caab$ and $caac$, having two distinct synchronizing states.

The right one restricted to a, b is the example found in [4] that we call CPR in Section 3. However, the extra non-trivial symbol c does not occur in any known DFA on four states with shortest synchronizing word length 9. It has eight distinct synchronizing words of the minimal length 9: in $baababaab$ the first, second and last b may be replaced by c , while replacement of the last b yields a distinct synchronizing state.

In the partial order on the 15 critical basic DFAs on three states, the four given in [16] are the minimal ones, but there is only one maximal one, being an upper bound of all. Here *maximal* means that it does not admit an extension that is still critical. In the partial order on the 12 critical basic DFAs on four states, the four given in [16] are the minimal ones, and exactly three are maximal, two of which being minimal, and the other is an upper bound of the two remaining minimal ones.

For $n \geq 5$ we assume Trahtman's conjecture, that is, apart from C_n there are only two DFAs with the minimality requirement having minimal synchronizing word length $(n - 1)^2$: one with 5 and one with 6 states. As a main result we prove that these are the only basic critical DFAs, so the phenomenon that we found for $n = 3, 4$ that a non-trivial extension of a critical DFA may be critical, does not occur for $n \geq 5$ assuming Trahtman's conjecture. For C_n this boils down to the main theorem stating that when adding an extra symbol to C_n not acting as the identity or as one of the existing symbols, always a strictly shorter synchronizing word can be obtained. The theorem is proved by a case analysis in how this extra symbol acts on the states.

This paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we consider small DFAs of at most six states. First we give a self-contained analysis of all DFAs on three and four states, satisfying our requirements. Next assuming Trahtman's conjecture, we do the same for DFAs on five and six states. The most substantial part is Section 4, where we prove our property for C_n for arbitrary n : C_n is maximal. This is done by assuming that

extending C_n by a symbol c is still critical. An extensive cases analysis on how c acts on the n states shows that this always yields a shorter synchronizing word. We conclude in Section ??.

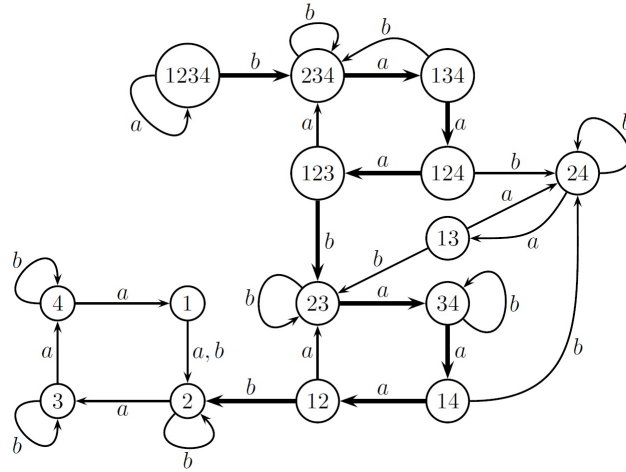
2 Preliminaries

A *deterministic finite automaton (DFA)* over a finite alphabet Σ consists of a finite set Q of states and a map $\delta : Q \times \Sigma \rightarrow Q$. Often the definition also involves an initial state and a set of final states, but for synchronization they may be ignored. For $w \in \Sigma^*$ and $q \in Q$ define qw inductively by $q\epsilon = q$ and $qwa = \delta(qw, a)$ for $a \in \Sigma$. So qw is the state where one ends when starting in q and applying δ -steps for the symbols in w consecutively, and qa is a short hand notation for $\delta(q, a)$. A word $w \in \Sigma^*$ is called *synchronizing* if a state $q_s \in Q$ exists such that $qw = q_s$ for all $q \in Q$. Stated in words: starting in any state q , after processing w one always ends in state q_s . Obviously, if w is a synchronizing word then so is wu for any word u . A DFA on n states is *critical* if its shortest synchronizing word has length $(n-1)^2$; it is *super-critical* if its shortest synchronizing word has length $> (n-1)^2$. A DFA is *extended* if one or more alphabet symbols are added. A critical DFA is *minimal* if it is not the extension of another critical DFA; it is *maximal* if it does not admit a critical extension.

The basic tool to analyze synchronization is by exploiting the *power set automaton*. For any DFA (Q, Σ, δ) its power set automaton is the DFA $(2^Q, \Sigma, \delta')$ where $\delta' : 2^Q \times \Sigma \rightarrow 2^Q$ is defined by

$$\delta'(V, a) = \{q \in Q \mid \exists p \in V : \delta(p, a) = q\}.$$

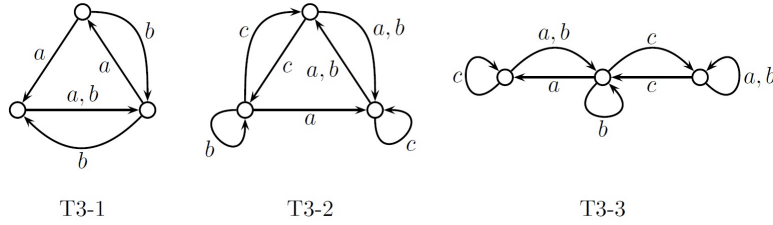
For any $V \subseteq Q, w \in \Sigma^*$ we define Vw as above, using δ' instead of δ . From this definition one easily proves that $Vw = \{qw \mid q \in V\}$ for any $V \subseteq Q, w \in \Sigma^*$. A set of the shape $\{q\}$ for $q \in Q$ is called a *singleton*. So a word w is synchronizing if and only if Qw is a singleton. Hence a DFA is synchronizing if and only if its power set automaton admits a path from Q to a singleton, and the shortest length of such a path corresponds to the shortest length of a synchronizing word. As an example we present the power set automaton of C_4 , in which indeed the unique shortest path from Q to a singleton (indicated by fat arrows from 1234 to 2) has length 9.



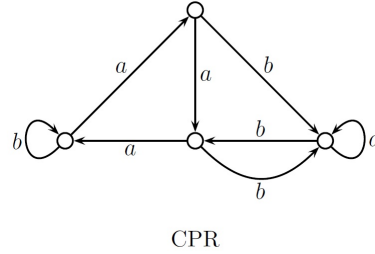
3 Small DFAs

3.1 Three and four states

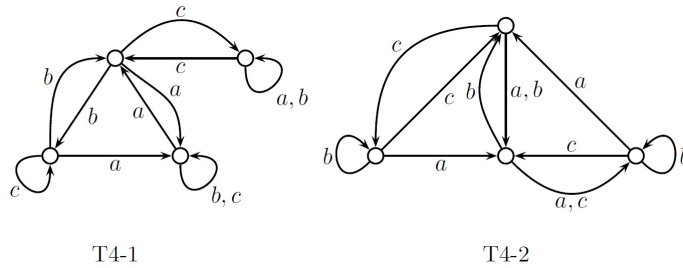
First we give the critical DFAs as presented in [16] on three and four states, apart from C_3 and C_4 , starting by three:



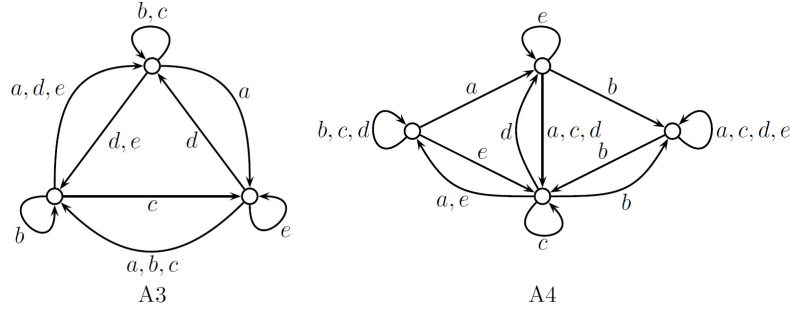
We call them T3-1, T3-2 and T3-3, as they were found by Trahtman. They all have a unique synchronizing word of length 4, being $baab$, $acba$, $bacb$, respectively. Next we give the three critical DFAs on four states apart from C_4 . The first one is CPR, found by Černý, Piricka and Rosenauerova, [4], and has unique synchronizing word of length 9, being $baababab$.



The next two we call T4-1 and T4-2, as they were found by Trahtman. They both have a unique synchronizing word of length 9, being $abcacabca$, $acbaaacba$, respectively.



In [16] it was investigated that together with C_3 and C_4 these are all minimal basic critical DFAs on $n = 3, 4$ with $q \leq 4$. Our goal is to complete the investigation by dropping both requirements of $q \leq 4$ and minimality. In order to do so we give two special DFAs with three and four states, called A3 and A4, both on five symbols a, b, c, d, e .



Theorem 1. *No supercritical DFAs on three states exist, and a basic DFA on three states is critical if and only if up to isomorphism it is one of the 15 automata that can be obtained from A3 by removing zero or more symbols and keeping at least one of the sets $\{a, b\}$, $\{a, d\}$, $\{b, c, e\}$, $\{c, d, e\}$ of symbols.*

Observe that A3 restricted to a, b coincides with C3, A3 restricted to a, d coincides with T3-1, A3 restricted to c, d, e coincides with T3-2 and A3 restricted to b, c, e coincides with T3-3, so exactly the minimal four critical automata on three states from [16]. On the other hand, as all basic critical DFAs on three states are contained in A3, A3 is the only maximal basic critical DFA on three states.

Proof. Let 1,2,3 be the three states. The automaton has a shortest synchronizing word of length ≥ 4 if and only if the shortest path from $\{1, 2, 3\}$ to a singleton in the power set automaton has length ≥ 4 . There is a step from $\{1, 2, 3\}$ to a smaller set. Since the length of the shortest path is ≥ 4 , this smaller set is not a singleton, so it is a pair; without loss of generality we may assume this is $\{2, 3\}$. Let b be the first symbol of a shortest synchronizing word, so $\{1, 2, 3\} \xrightarrow{b} \{2, 3\}$. Since the shortest path from $\{2, 3\}$ to a singleton consists of at least three steps, it meets the other two pairs and consists of exactly three steps, yielding shortest synchronizing word length 4. May be after swapping 2 and 3 we may assume this shortest path is

$$\{1, 2, 3\} \xrightarrow{b} \{2, 3\} \rightarrow \{1, 2\} \rightarrow \{1, 3\} \rightarrow \text{singleton}.$$

As it is the shortest path, we conclude that for every symbol a we have

- either $\{1, 2, 3\} \xrightarrow{a} \{1, 2, 3\}$ or $\{1, 2, 3\} \xrightarrow{a} \{2, 3\}$,
- either $\{2, 3\} \xrightarrow{a} \{2, 3\}$ or $\{2, 3\} \xrightarrow{a} \{1, 2\}$, and
- not $\{1, 2\} \xrightarrow{a} \text{singleton}$.

A small program investigates that among the $3^3 = 27$ possible symbol actions in a DFA on three states exactly 6 satisfy these properties: exactly the symbols a, b, c, d, e, f in A3. So for all DFAs being a sub-automaton of A3 it hold that if it is synchronizing, then the shortest synchronizing word length is 4. Restricting A3 to either $\{a, b\}$, $\{a, d\}$, $\{b, c, e\}$ or $\{c, d, e\}$ yields one of the known synchronizing

DFA's, so every supersystem is synchronizing too. Conversely, it is easily checked that every strict subsystem of these is not synchronizing. This concludes the proof. \square

For an automaton A write A^+ for A to which a fresh symbol f is added for which $qf = q$ for all states q .

Theorem 2. *No supercritical DFA's on four states exist, and a basic DFA on three states is critical if and only if up to isomorphism it is C_4 , T_4-2 , or one of the 10 automata that can be obtained from A_4 by removing zero or more symbols and keeping at least one of the sets $\{a, b\}$, $\{b, d, e\}$ of symbols.*

Observe that A_4 restricted to a, b coincides with CPR and A_4 restricted to b, d, e coincides with T_4-1 , so together with C_4 and T_4-2 exactly the four automata with four states from [16], being the minimal ones. On the other hand, an immediate consequence of this theorem is that C_4 , T_4-2 and A_4 are the only maximal basic critical DFA's on four states.

Proof. Let $1, 2, 3, 4$ be the four states. We have to prove that the shortest path in the power set automaton from $\{1, 2, 3, 4\}$ to a singleton never has length > 9 (this would be super-critical), and that length 9 only occurs in the cases indicated by the theorem. So assume this length is ≥ 9 . Since there is a step from $\{1, 2, 3, 4\}$ to a smaller set, and since the length is ≥ 9 , it is not to a pair since there are only 6 distinct pairs. So after one step only one element is removed from $\{1, 2, 3, 4\}$, say, 4. By the same argument also the next step in a shortest path to a singleton is not to a pair; by possibly renaming we may assume it is to $\{2, 3, 4\}$, so a shortest path is of the shape

$$\{1, 2, 3, 4\} \xrightarrow{a_1} \{1, 2, 3\} \xrightarrow{a_2} \{2, 3, 4\} \rightarrow^{\geq 7} \text{singleton}.$$

The approach is to generate all solutions by a computer program. In order to reduce the search space, first we prove that for every symbol a we have

1. if $\{1, 2, 3, 4\} \xrightarrow{a} V$ then $\{1, 2, 3\} \subseteq V$;
2. if $\{1, 2, 3\} \xrightarrow{a} V$ then either $V = \{1, 2, 3\}$ or $V = \{2, 3, 4\}$;
3. if $\{2, 3, 4\} \xrightarrow{a} V$ then $|V| = 3$.

We start by property 3. Assume that a symbol a exists such that $\{2, 3, 4\} \xrightarrow{a} \{p_1, q_1\}$. As the DFA is synchronizing, there is a path from $\{p_1, q_1\}$ to a singleton in the power set automaton. Take a shortest such path. As the shortest path from $\{2, 3, 4\}$ to a singleton consists of at least 7 steps, the shortest path from $\{p_1, q_1\}$ to a singleton consists of at least 6 steps. As there are only 6 distinct unordered pairs, it can not be longer than 6 steps, and this shortest path is of the shape

$$\{p_1, q_1\} \rightarrow \{p_2, q_2\} \rightarrow \{p_3, q_3\} \rightarrow \{p_4, q_4\} \rightarrow \{p_5, q_5\} \rightarrow \{p_6, q_6\} \rightarrow \text{singleton},$$

in which $\{p_i, q_i\}$ are the six distinct unordered pairs, together with the starting part $\{1, 2, 3, 4\} \xrightarrow{a_1} \{1, 2, 3\} \xrightarrow{a_2} \{2, 3, 4\} \xrightarrow{a} \{p_1, q_1\}$ yielding a shortest synchronizing sequence of length 9. From this pattern we want to derive a contradiction.

In principal this could be done by hand using a lot of case analysis. In order to avoid this, and as in circumstances like these computers may be more reliable than humans, we chose to do this by computer support. We built a formula on eight unknown functions $a_i : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ for $i = 1$ to 8, and 12 values $p_i, q_i \in \{1, 2, 3, 4\}$ for $i = 1$ to 6, stating that

- $a_1(\{1, 2, 3, 4\}) = \{1, 2, 3\}$,
- $a_2(\{1, 2, 3\}) = \{2, 3, 4\}$,
- $a_3(\{2, 3, 4\}) = \{p_1, q_1\}$,
- $a_{i+3}(\{p_i, q_i\}) = \{p_{i+1}, q_{i+1}\}$ for $i = 1, 2, 3, 4, 5$,
- $p_i \neq q_i$ for $i = 1, 2, 3, 4, 5, 6$,
- $\{p_i, q_i\} \neq \{p_j, q_j\}$ for $1 \leq i < j \leq 6$,
- $a_i(p_j) \neq a_i(q_j)$ for all $i = 1, 2, 3, 4, 5, 6$, $j = 1, 2, 3, 4, 5$, and
- $a_k(\{p_i, q_i\}) \neq \{p_j, q_j\}$ for $i + 1 < j \leq 6$ and $k = 1, \dots, 8$.

The last two requirements may be stated since otherwise a shorter path to a singleton can be obtained. Next we applied the SMT solver Yices to this formula, that stated that this formula is unsatisfiable in a fraction of a second. So this yields the required contradiction, proving property 3. To check that we generated the correct formulas, we also applied the SMT solver on variants of the formula in which minor parts of the formula were removed, and the obtained satisfying assignments yielded solutions of the modified problem that could be checked by hand.

In fact we proved that in the power set automaton no pair can be reached from $\{1, 2, 3, 4\}$ in less than three steps.

In order to prove properties 1 and 2, observe that from $a_1(\{1, 2, 3, 4\}) = \{1, 2, 3\}$ we conclude that $x \neq y \in \{1, 2, 3, 4\}$ exist such that $a_1(x) = a_1(y)$. Since $a_1(\{2, 3, 4\})$ consists of three elements by property 3, we have $\{x, y\} \not\subseteq \{2, 3, 4\}$, so $1 \in \{x, y\}$. If $a_1(\{1, 2, 3\})$ consists of two elements this gives rise to a shorter synchronizing sequence, so $\{x, y\} \not\subseteq \{1, 2, 3\}$, hence $4 \in \{x, y\}$. So $a_1(1) = a_1(4)$.

For property 1 assume that $\{1, 2, 3\} \not\subseteq V$ for $V = a(\{1, 2, 3, 4\})$. Since $|V| = 3$, the set V is one of the three sets $\{2, 3, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$. If it is $\{2, 3, 4\}$, we have a shorter synchronizing sequence; otherwise $\{1, 4\} \subseteq V$, by which $|a_1(V)| = 2$, and this pair $a_1(V)$ can be reached in two steps a, a_1 from $\{1, 2, 3, 4\}$. Both cases yield a contradiction, proving property 1.

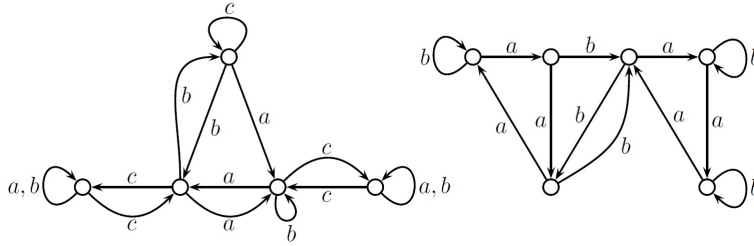
The proof of property 2 is similar: if $V = a(\{1, 2, 3\})$ is not $\{1, 2, 3\}$ or $\{2, 3, 4\}$, then by $|V| = 3$ we have V is $\{1, 2, 4\}$ or $\{1, 3, 4\}$, so $|a_1(V)| = 2$, and this pair $a_1(V)$ can be reached in two steps a, a_1 from $\{1, 2, 3, 4\}$, contradiction.

It turns out that among the 256 functions from $\{1, 2, 3, 4\}$ to itself exactly 18 satisfy properties 1, 2 and 3. This includes the identity that has to be excluded since that is not allowed in a basic DFA, leaving 17 functions. So our computer search may restrict to basic DFAs on four states such that all symbols act as one of these 17 functions. There are $2^{17} = 131072$ of them. Our program generates all these, and computes for all of them the power automaton and checks whether it admits a path from $\{1, 2, 3, 4\}$ to a singleton, and if so, what is the length of the shortest such path. If this length is ≥ 9 , the DFA is reported. This full computation is executed in less than 10 seconds. As expected, no shortest

path length longer than 9 is obtained, proving that no supercritical DFA exists. Exactly 24 basic DFAs are obtained with shortest path length 9. These 24 automata exactly coincide with the 12 automata indicated in the theorem; each occurring twice up to swapping 2 and 3. \square

3.2 Five and six states

In Section 3 we saw that for $n = 3, 4$ on n states a DFA with shortest synchronizing word length $(n - 1)^2$ may have extensions keeping the same property, in another way than just copying existing symbols or adding a symbol that acts as the identity. We now claim that for $n \geq 5$ this does not occur any more, assuming Trahtman's conjecture. In Section 4 we will prove that this holds for C_n for all $n \geq 3$. By Trahtman's conjecture the only two more DFAs to consider are one on five states and one on six states. The one on five states is from Roman [13], the one on six from Kari [9]. They are depicted as follows.



For $n = 5, 6$ we wrote a program that takes a DFA on n states and computes for all n^n ways to add a fresh symbol, the shortest path length in the power set automaton from the full set to a singleton. For both candidates it turns out that the only extensions keeping this shortest path length to be $(n - 1)^2$ is by adding either a copy of one of the existing symbols, or a symbol that acts as the identity. This proves our claim.

In fact we also applied this approach to $n = 3, 4$. Our results Theorem 1 and Theorem 2 were first found in this way assuming Trahtman's conjecture; the approach presented in Section 3 was found later, and is preferred since it does not assume Trahtman's conjecture.

4 Extending C_n

In this section we show that for all $n \geq 5$ the DFA C_n is maximal: it cannot be extended to a basic DFA with the same synchronizing length. The main result of this section is the following:

Theorem 3. *Let $n \geq 5$ and let C_n^c be a basic extension of C_n by a symbol c . Then C_n^c admits a synchronizing word of length strictly less than $(n - 1)^2$.*

Recall that *basic* means that c is not equal to a or b and that c is not the identity function on Q . This section is organized as follows: first we collect some properties of C_n and its unique shortest synchronizing word. Then we consider the cases $|Qc| = n$, $|Qc| = n - 1$ and $|Qc| \leq n - 2$ separately.

4.1 Properties of C_n

Recall that C_n is defined by n states $1, 2, \dots, n$, and two symbols a, b , acting by $qa = q + 1$ for $q = 1, \dots, n - 1$, $na = 1$, and $qb = q$ for $q = 2, \dots, n$, $1b = 2$. It is well known that C_n has the following shortest synchronizing word:

$$w_n = b(a^{n-1}b)^{n-2},$$

which has length $|w_n| = 1 + n(n - 2) = (n - 1)^2$. This word is synchronizing since

$$Qb = \{2, 3, \dots, n\} \quad (1)$$

$$\{2, 3, \dots, k\} a^{n-1}b = \{2, 3, \dots, k - 1\}, \quad 3 \leq k \leq n. \quad (2)$$

The first part of this word defines the path

$$Q \xrightarrow{b} Q \setminus \{1\} \xrightarrow{a} Q \setminus \{2\} \xrightarrow{a} \dots \xrightarrow{a} Q \setminus \{n\}. \quad (3)$$

We now extend the alphabet of the automaton by a non-trivial new symbol c . Non-trivial means that the transitions defined by c are not all equal to the transitions of a or the transitions of b and furthermore that c is not the identity function. We will distinguish three cases:

1. $|Qc| = n$, i.e. c is a permutation.
2. $|Qc| = n - 1$, i.e. c has deficiency 1.
3. $|Qc| \leq n - 2$, i.e. c has deficiency 2.

We will show that in all these cases a shorter synchronizing word exists. The general pattern in the arguments is as follows. The shortest synchronizing word w_n corresponds to a path from Q to a singleton in the power automaton of C_n . Take two sets $S, S' \subseteq Q$ on this path which are visited in this order. Let d be the distance from S to S' , i.e.

$$d := \min \{|w| : Sw = S', w \in \{a, b\}^*\}.$$

Now construct a word $w \in \{a, b, c\}^*$ in the automaton C_n^c for which $Sw = S'$ and $|w| < d$. Then C_n^c admits a synchronizing word of length at most $|w_n| - d + |w| < (n - 1)^2$.

4.2 The additional symbol defines a permutation

If c defines a permutation on Q , we may assume that c satisfies:

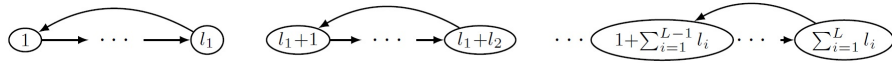
$$qc \leq q + 1 \text{ for all } q \in Q. \quad (4)$$

Indeed, if $qc = q + k$ for some $q \in Q$ and $k \geq 2$, then $(Q \setminus \{q\})c = Q \setminus \{q + k\}$, which in view of (3) would imply existence of a synchronizing word shorter than $(n - 1)^2$. The following lemma describes the structure of c .

Lemma 1. *If $|Q| = n \geq 1$ and c is a permutation on Q satisfying (4), then there exist numbers L (number of c -loops) and $1 \leq l_1, \dots, l_L \leq n$ (lengths of c -loops) with $\sum_{i=1}^L l_i = n$ such that*

$$qc = \begin{cases} q - l_i + 1 & \text{if } q = l_1 + \dots + l_i \text{ for some } 1 \leq i \leq L \\ q + 1 & \text{otherwise} \end{cases} \quad (5)$$

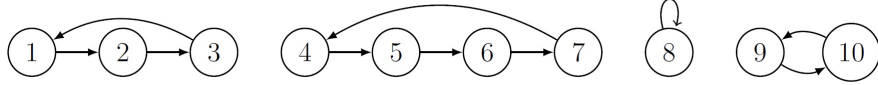
An illustration of the statement is given below.



Proof. We give a proof by induction. For $n = 1$, $1 \xrightarrow{c} 1$, so $L = 1$ and $l_1 = 1$.

Now suppose the statement is true for all $n \leq N$ and consider the case $|Q| = N + 1$. If $1 \xrightarrow{c} 1$, then c defines a permutation on $Q \setminus \{1\}$. Applying the induction hypothesis on $Q \setminus \{1\}$ gives the result. If $1 \xrightarrow{c} 2 \xrightarrow{c} \dots \xrightarrow{c} k$ for some $k \geq 2$, then either $kc = k + 1$ or $kc = 1$. In both cases there is a number $l_1 \geq 1$ such that $1 \xrightarrow{c} \dots \xrightarrow{c} l_1 \xrightarrow{c} 1$. Apply the induction hypothesis on the remaining $n - l_1$ states. \square

Note that $L = 1$ and $L = n$ are the trivial cases, because then $c = a$ or c is the identity. Before we give a general argument, we first give an example.



Example 1. Consider the automaton $C_{10}^c = \{Q, \Sigma, \delta\}$ with $Q = \{1, \dots, 10\}$ and $\Sigma = \{a, b, c\}$. The actions of the symbols a and b are from the definition of C_n and c is the permutation shown above. Here we have four loops ($L = 4$) with lengths $l_1 = 3, l_2 = 4, l_3 = 1$ and $l_4 = 2$. We will show how to use the c -loop of length four to create a shorter synchronizing word. Consider the set

$$S = \{2, \dots, 9\}$$

We start by a greedy approach to reach a set of size 7:

$$Sa^3b = (\{1, 2\} \cup \{5, \dots, 10\})b = \{2\} \cup \{5, \dots, 10\}.$$

As a next step, we shift everything by using the symbol a until the isolated state $\{2\}$ ends up in the c -loop of length four:

$$(\{2\} \cup \{5, \dots, 10\})a^3 = \{1, 2, 3\} \cup \{5\} \cup \{8, 9, 10\}$$

Since $\{1, 2, 3\}$ and $\{8, 9, 10\}$ are (unions of) full c -loops, they are invariant under c . Therefore, we can move the isolated state $\{5\}$ to the desired position:

$$(\{1, 2, 3\} \cup \{5\} \cup \{8, 9, 10\})c^3 = \{1, 2, 3, 4\} \cup \{8, 9, 10\}$$

Finally, we shift again by a power of a and apply b to get rid of one more state:

$$(\{1, 2, 3, 4\} \cup \{8, 9, 10\}) a^3 b = \{1, \dots, 7\} b = \{2, \dots, 7\} := S'.$$

We conclude that the word $w = a^3 b a^3 c^3 a^3 b$ has the property that $Sw = S'$. In C_{10} both S and S' are on the shortest path from Q to $\{2\}$ and by (2) the distance between them is equal to $2n = 20$. The word w has length $|w| = 14$, so in C_{10}^c there exists a synchronizing word of length at most $(10 - 1)^2 - 6 = 75$. Note that there might be even shorter synchronizing words, but for our main goal it is sufficient to have some synchronizing word shorter than 81.

The idea of this example works in more generality if there is a c -loop of length at least 3, as is proved in the next lemma. If the longest loop has length 2, then basically we can do the same thing, but we need at least three c -loops to isolate a state.

Lemma 2. *Let $n \geq 5$ and let C_n^c be an extension of the automaton C_n by a symbol c as given in Lemma 1. If $2 \leq L \leq n - 1$, then C_n^c admits a synchronizing word of length strictly less than $(n - 1)^2$.*

Proof. We distinguish the following three cases:

- $L \geq 2$ and $l_k \geq 3$ for some k .
- $L \geq 3$ and $l_k = 2$ for some $k \leq L - 1$.
- $L \geq 3$ and $l_L = 2$.

Note that for all $n \geq 5$ and all possible non-trivial choices of c , the extended automaton C_n^+ satisfies at least one of these cases.

Case 1: $L \geq 2$ and $l_k \geq 3$ for some k . Take k such that $l_k \geq 3$ and write

$$\Lambda^- = \sum_{i=1}^{k-1} l_i, \quad \Lambda^+ = \sum_{i=k+1}^L l_i,$$

for the sum of the looplengths before the k th loop and after the k th loop respectively. These sums can be zero if $k = 1$ or $k = L$. Define $\Lambda = \Lambda^- + \Lambda^+ = n - l_k \leq n - 3$. Since $L \geq 2$, we have $\Lambda \geq 1$. Take

$$S = \{2, 3, \dots, n - l_k + 3\}, \quad S' = \{2, 3, \dots, n - l_k + 1\}.$$

and define the word

$$w = a^{l_k-1} b a^{\Lambda^-} c^{l_k-1} a^{\Lambda^+} b. \tag{6}$$

We will show that $Sw = S'$. Write $S = S_1 \cup S_2$ with

$$\begin{aligned} S_1 &= \{2, \dots, n - l_k + 1\} = \{2, \dots, 1 + \Lambda\}, \\ S_2 &= \{n - l_k + 2, n - l_k + 3\} = \{2 + \Lambda, 3 + \Lambda\}. \end{aligned}$$

Then

$$\begin{aligned}
S_1 w &= \{2, \dots, 1 + \Lambda\} a^{l_k-1} b a^{\Lambda^-} c^{l_k-1} a^{\Lambda^+} b \\
&= \{l_k + 1, \dots, n\} b a^{\Lambda^-} c^{l_k-1} a^{\Lambda^+} b \\
&= \{l_k + 1, \dots, n\} a^{\Lambda^-} c^{l_k-1} a^{\Lambda^+} b \\
&= (\{1, \dots, \Lambda^-\} \cup \{\Lambda^- + l_k + 1, \dots, n\}) c^{l_k-1} a^{\Lambda^+} b \\
&= (\{1, \dots, \Lambda^-\} \cup \{\Lambda^- + l_k + 1, \dots, n\}) a^{\Lambda^+} b \\
&= \{1, \dots, \Lambda\} b \\
&= \begin{cases} \{2\} = \{1 + \Lambda\} & \text{if } \Lambda = 1 \\ \{2, \dots, \Lambda\} & \text{if } \Lambda \geq 2, \end{cases}
\end{aligned} \tag{7}$$

where sets of the form $\{x, \dots, y\}$ with $x > y$ should be interpreted as being empty. This occurs if $\Lambda^- = 0$ or $\Lambda^+ = 0$. Furthermore

$$\begin{aligned}
S_2 w &= \{2 + \Lambda, 3 + \Lambda\} a^{l_k-1} b a^{\Lambda^-} c^{l_k-1} a^{\Lambda^+} b \\
&= \{1, 2\} b a^{\Lambda^-} c^{l_k-1} a^{\Lambda^+} b = \{2\} a^{\Lambda^-} c^{l_k-1} a^{\Lambda^+} b \\
&= \{2 + \Lambda^-\} c^{l_k-1} a^{\Lambda^+} b = \{1 + \Lambda^-\} a^{\Lambda^+} b \\
&= \{1 + \Lambda\} b = \{1 + \Lambda\}.
\end{aligned} \tag{8}$$

It follows that the word w has the property

$$Sw = (S_1 \cup S_2)w = S_1 w \cup S_2 w = \{2, \dots, \Lambda + 1\} = S'.$$

and its length is

$$|w| = l_k - 1 + 1 + \Lambda^- + l_k - 1 + \Lambda^+ + 1 = 2l_k + \Lambda = l_k + n < 2n.$$

In the automaton C_n the sets S and S' are both on the shortest path from Q to a singleton and the shortest path is defined by $S(a^{n-1}b)^2 = S'$. Since $|(a^{n-1}b)^2| = 2n > |w|$, the statement of the lemma follows.

The above proof fails in case $l_k \leq 2$, since then $n - l_k + 3 > n$. However, the proofs for the other cases use pretty much the same ideas.

Case 2: $L \geq 3$ and $l_k = 2$ for some $k \leq L - 1$. Take k such that $l_k = 2$ and write

$$\Lambda^- = \sum_{i=1}^{k-1} l_i, \quad \Lambda^+ = \sum_{i=k+2}^L l_i,$$

for the sum of the looplengths before the k th loop and after the $(k + 1)$ th loop respectively. These sums can be zero if $k = 1$ or $k = L - 1$. Define $\Lambda = \Lambda^- + \Lambda^+ = n - l_k - l_{k+1} \leq n - 3$. From the assumption $L \geq 3$ it follows that $\Lambda \geq 1$. Take

$$S = \{2, 3, \dots, \Lambda + 3\}, \quad S' = \{2, 3, \dots, \Lambda + 1\}.$$

and define the word

$$w = a^{l_k+l_{k+1}-1}ba^{A^-}ca^{A^+}b.$$

By a similar argument as in Case 1 it follows that $Sw = S'$: Let $S_1 = \{2, \dots, A+1\}$, then

$$\begin{aligned} S_1w &= \{2, \dots, A+1\} a^{l_k+l_{k+1}-1}ba^{A^-}ca^{A^+}b \\ &= \{l_k+l_{k+1}+1, \dots, n\} ba^{A^-}ca^{A^+}b \\ &= \{l_k+l_{k+1}+1, \dots, n\} a^{A^-}ca^{A^+}b \\ &= (\{1, \dots, A^-\} \cup \{A^-+l_k+l_{k+1}+1, \dots, n\}) ca^{A^+}b \\ &= (\{1, \dots, A^-\} \cup \{A^-+l_k+l_{k+1}+1, \dots, n\}) a^{A^+}b \\ &= \{1, \dots, A\} b = \begin{cases} \{2\} = \{1+A\} & \text{if } A = 1 \\ \{2, \dots, A\} & \text{if } A \geq 2, \end{cases} \end{aligned} \quad (9)$$

Completely analogous to Case 1, we have

$$\{A+2, A+3\}w = \{1+A\}.$$

Therefore,

$$Sw = \{2, \dots, A+1\}w \cup \{A+2, A+3\}w = \{2, \dots, A+1\} = S'$$

Since w has length $n+2 < 2n$, the statement of the lemma follows.

Case 3: $L \geq 3$ and $l_L = 2$. Define

$$S = \{2, \dots, n\}, \quad w = a^2ba^{n-3}cab. \quad (10)$$

Then

$$\begin{aligned} Sw &= (\{1, 2\} \cup \{4, \dots, n\})ba^{n-3}cab = (\{2\} \cup \{4, \dots, n\})a^{n-3}cab \\ &= (\{n-1\} \cup \{1, \dots, n-3\})cab = (\{n\} \cup \{1, \dots, n-3\})ab \\ &= \{1, \dots, n-2\}b = \{2, \dots, n-2\}. \end{aligned} \quad (11)$$

Since $|w| = n+3 < 2n$, the result follows. \square

4.3 The additional symbol has deficiency 1

In this section we assume that the additional symbol c satisfies $|Qc| = n-1$. We will prove that the extended automaton C_n^c admits a synchronizing word of length strictly less than $(n-1)^2$ for every non-trivial choice of c . The first step (Lemma's 3, 4, 5 and Corollary 1) is to show that the only candidates to preserve the shortest synchronizing word length have a loop structure similar to the permutations in Lemma 1. In Lemma 6 we couple such candidates c to a permutation \tilde{c} , which leads to the conclusion that the automaton with c synchronizes at least as fast as the automaton with \tilde{c} .

Lemma 3. *Let $n \geq 5$ and let C_n^c be an extension of the automaton C_n by a symbol c for which $|Qc| = n - 1$. If the shortest synchronizing word for C_n^c has length $(n - 1)^2$, then $Qc = Q \setminus \{1\}$ and c defines a permutation on $Q \setminus \{1\}$.*

Proof. If $Qc = Q \setminus \{q\}$ with $q \neq 1$, then $w = ca^{n-q}b(a^{n-1}b)^{n-3}$ is synchronizing and w has length

$$|w| = 1 + n - q + 1 + n(n - 3) = (n - 1)^2 - q + 1 < (n - 1)^2.$$

If $Qc = Q \setminus \{1\}$ and $|Qc^2| \leq n - 2$, then one of the following two is true:

– $Qc^2 = Q \setminus \{1, 2\}$.

In this case $w = c^2a^{n-2}b(a^{n-1}b)^{n-4}$ is synchronizing and has length

$$|w| = 2 + n - 2 + 1 + n(n - 4) = (n - 1)^2 - n < (n - 1)^2.$$

– $Qc^2 \subset Q \setminus \{q\}$ for some $q \geq 3$.

In this case $w = c^2a^{n-q}b(a^{n-1}b)^{n-3}$ is synchronizing and w has length

$$|w| = 2 + n - q + 1 + n(n - 3) = (n - 1)^2 - q + 2 < (n - 1)^2.$$

Therefore, we may assume that $Qc = Q \setminus \{1\}$ and $|Qc^2| = n - 1$. This means that $(Q \setminus \{1\})c = Q \setminus \{1\}$, so c defines a permutation on $Q \setminus \{1\}$. \square

The next lemma shows that c can be assumed to satisfy $qc \leq q + 1$ for all q .

Lemma 4. *Let $n \geq 5$ and let C_n^c be an extension of the automaton C_n by a symbol c for which $|Qc| = n - 1$. If $qc = q + k$ for some $q \in Q$ and $k \geq 2$, then C_n^c admits a synchronizing word of length strictly less than $(n - 1)^2$.*

Proof. If $qc = q + k$ for some $q \in Q$ and $k \geq 2$, then either $1c \neq 2$ or $qc = q + k$ for some $q \geq 2$ and $k \geq 2$. We distinguish these two cases:

– $1c \neq 2$. In this case there exists a singleton $\tilde{q} := 2c^{-1}$, so

$$(Q \setminus \{\tilde{q}\})c = Q \setminus \{1, 2\}.$$

The sets $Q \setminus \{\tilde{q}\}$ and $Q \setminus \{1, 2\}$ are both on the shortest path in C_n , where

$$(Q \setminus \{\tilde{q}\})a^{n-\tilde{q}}ba = Q \setminus \{1, 2\}.$$

Since $a^{n-\tilde{q}}ba \geq 2$, the shortest synchronizing word in C_n^c has length at most $(n - 1)^2 - 1$.

– $1c = 2$ and there exist $q \geq 2$ and $k \geq 2$ such that $qc = q + k$. In this case

$$(Q \setminus \{q\})c = Q \setminus \{1, q + k\} \subseteq Q \setminus \{q + k\},$$

which means that there is synchronizing word of length $(n - 1)^2 - k + 1$ in C_n^c , see (3).

□

Lemma 5. Suppose $|Q| = n \geq 2$ and c is such that

$$Qc = Q \setminus \{1\}, \quad (Q \setminus \{1\})c = Q \setminus \{1\} \quad \text{and} \quad qc \leq q + 1 \text{ for all } q. \quad (12)$$

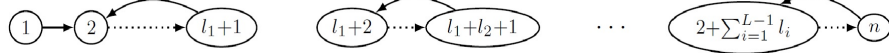
Then there exist numbers L (number of c -loops) and $1 \leq l_1, \dots, l_L \leq n - 1$ (lengths of c -loops) with $\sum_{i=1}^L l_i = n - 1$ such that

$$qc = \begin{cases} q - l_i + 1 & \text{if } q = l_1 + \dots + l_i + 1 \text{ for some } 1 \leq i \leq L \\ q + 1 & \text{otherwise} \end{cases} \quad (13)$$

Proof. Similar to the proof of Lemma 1.

Corollary 1. Let $n \geq 5$ and let C_n^c be an extension of the automaton C_n by a symbol c for which $|Qc| = n - 1$. If the shortest synchronizing word for C_n^c has length $(n - 1)^2$, then c has the structure described in Lemma 5.

An illustration of the statement is given below. The structure of c if $|Qc| = n - 1$. Dotted arrows represent chains of transitions of the form $qc = q + 1$.



Finally, in the next lemma, we handle symbols c having the structure described in Lemma 5. If all loops of c have length 1, then $qc = qb$ for all $q \in Q$. Therefore the case $L = n - 1$ is excluded.

Lemma 6. Let $n \geq 5$ and let C_n^c be an extension of the automaton C_n by a symbol c as given in Lemma 5. If $1 \leq L \leq n - 2$, then C_n^c admits a synchronizing word of length strictly less than $(n - 1)^2$.

Proof. We distinguish two cases: $2 \leq l_1 \leq n - 1$ and $l_1 = 1$.

Case 1: $2 \leq l_1 \leq n - 1$. In this case

$$Q \setminus \{l_1\} \xrightarrow{c} Q \setminus \{1, l_1 + 1\} \xrightarrow{c} Q \setminus \{1, 2\}.$$

In C_n the shortest path between these sets is given by

$$Q \setminus \{l_1\} \xrightarrow{a^{n-l_1}} Q \setminus \{n\} \xrightarrow{b} Q \setminus \{1, n\} \xrightarrow{a} Q \setminus \{1, 2\},$$

which has length $n - l_1 + 2 \geq 3$. Therefore, C_n^c has a synchronizing word of length at most $(n - 1)^2 - 1$.

Case 2: $l_1 = 1$. This means that $2c = 2$. We define a permutation \tilde{c} on Q by

$$q\tilde{c} = \begin{cases} 1 & \text{if } q = 1, \\ qc & \text{if } q \neq 1. \end{cases}$$

The permutation \tilde{c} has $\tilde{L} := L + 1 \geq 2$ loops and L of them coincide with the loops of c . Since c has a loop of length at least 2, so does \tilde{c} . The loop lengths of \tilde{c} are given by $\tilde{l}_1 = 1$ and $\tilde{l}_k = l_{k-1}$ for $2 \leq k \leq \tilde{L}$.

By Lemma 2 we already know that there exists a synchronizing word $\tilde{w} \in \{a, b, \tilde{c}\}^*$ with $|\tilde{w}| < (n-1)^2$. Define $w \in \{a, b, c\}^*$ as the word that is obtained from \tilde{w} by replacing all instances of \tilde{c} by c . Clearly this operation preserves the word length. We will show that the word w is a synchronizing word for C_n^c .

The key observation is that the permutation \tilde{c} has the following property for $S \subseteq Q$:

$$\text{If } 1 \notin S \text{ or } 2 \in S, \text{ then } Sc^k \subseteq S\tilde{c}^k \text{ for all } k \geq 1. \quad (14)$$

We consider the same cases as in the proof of Lemma 2:

– $\tilde{L} \geq 2$ and $\tilde{l}_k \geq 3$ for some k . In this case

$$\tilde{w} = a^{\tilde{l}_k-1} b a^{\Lambda^-} \tilde{c}^{\tilde{l}_k-1} a^{\Lambda^+} b$$

is synchronizing (compare to (6)), where

$$\Lambda^- = \sum_{i=1}^{k-1} \tilde{l}_i \geq \tilde{l}_1 + \tilde{l}_2 = 2, \quad \Lambda^+ = \sum_{i=k+1}^{\tilde{L}} \tilde{l}_i.$$

Here we used that $\tilde{l}_1 = \tilde{l}_2 = 1$ and $k \geq 3$ since $1\tilde{c} = 1$ and $2\tilde{c} = 2$. Let

$$T_1 = \{1, \dots, \Lambda^-\} \cup \{\Lambda^- + l_k + 1, \dots, n\} \quad \text{and} \quad T_2 = \{2 + \Lambda^-\},$$

and observe that $2 \in T_1$ and $1 \notin T_2$. By property (14), we obtain

$$T_1 c^{\tilde{l}_k-1} \subseteq T_1 \tilde{c}^{\tilde{l}_k-1}, \quad T_2 c^{\tilde{l}_k-1} \subseteq T_2 \tilde{c}^{\tilde{l}_k-1}.$$

Comparing with the argument in the proof of Lemma 2, in particular (7) and (8), we conclude that $Qw \subseteq Q\tilde{w}$ and w is synchronizing.

– $\tilde{L} \geq 3$ and $\tilde{l}_k = 2$ for some $k \leq \tilde{L} - 1$. Here an analogous argument as in the previous case gives the result.

– $\tilde{L} \geq 3$ and $\tilde{l}_{\tilde{L}} = 2$. Let

$$\tilde{w} = a^2 b a^{n-3} \tilde{c} a b,$$

analogous to (10). Since $n \geq 5$, we have

$$2 \in \{n-1\} \cup \{1, \dots, n-3\}.$$

Applying property (14) again, we obtain

$$(\{n-1\} \cup \{1, \dots, n-3\})c \subseteq (\{n-1\} \cup \{1, \dots, n-3\})\tilde{c}.$$

By comparing with (11), it follows that $Qw \subseteq Q\tilde{w}$ and therefore w synchronizes.

□

4.4 The additional symbol has deficiency at least 2

Lemma 7. *Let $n \geq 5$ and let C_n^c be an extension of the automaton C_n by a symbol c such that $|Qc| \leq n - 2$. Then C_n^c admits a synchronizing word of length strictly less than $(n - 1)^2$.*

Proof. There exists $q \geq 2$ such that $Qc \subset Q \setminus \{q\}$, which implies the result. \square

Proof of Theorem 3. Combining all results of the preceding sections completes the proof. \square

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